ON INTEGRABLE STRUCTURES FOR A GENERALIZED MONGE-AMPÈRE EQUATION

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ABSTRACT. We consider a 3rd-order generalized Monge-Ampère equation $u_{yyy} - u_{xxy}^2 + u_{xxx}u_{xyy} = 0$ (which is closely related to the associativity equation in the 2-d topological field theory) and describe all integrable structures related to it (i.e., Hamiltonian, symplectic, and recursion operators). Infinite hierarchies of symmetries and conservation laws are constructed as well.

INTRODUCTION

Monge-Ampère equations [13] is one of the most interesting objects to apply methods of geometrical theory of differential equation. Generalizations of classical Monge-Ampère equations are discussed, e.g., in [2]. One of such generalizations is the equation

$$u_{yyy} - u_{xxy}^2 + u_{xxx}u_{xyy} = 0. \tag{1}$$

This is a third-order Monge-Ampère equation ([2, 13]), but this does not help too much in understanding its integrability properties.

Equation (1) is closely related to the associativity equation in 2-d topological field theory [4] and was studied in a number of papers ([5, 6, 8, 9, 14]) and its integrability (existence of a bi-Hamiltonian structure) was established.

Note though that in these papers the equation was not considered in the initial form (1), but was rewritten as a three-component system

$$a_y = b_x, \quad b_y = c_x, \quad c_y = (b^2 - ac)_x \tag{2}$$

of hydrodynamical type. Of course, equations (1) and (2) are closely related, but not the same and even not equivalent being associated to each other by the differential substitution

$$a = u_{xxx}, \qquad b = u_{xxy}, \qquad c = u_{xyy}$$

(just like the KdV and mKdV are related by the Miura map or the Burgers and heat equations by the Cole-Hopf transformation).

The aim of this paper is to attack Equation (1) directly, not reducing it to the evolutionary form, and to study the structures that arise on this equation. To this end, we use geometrical and cohomological methods described initially in [10] and discussed in detail in a recent review paper [12].

Key words and phrases. Monge-Ampère equations, integrability, Hamiltonian operators, symplectic structures, symmetries, conservation laws, jet spaces, WDVV equations, 2-d topological field theory.

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These methods has been successfully applied to a number of equation (see, e.g., [7, 11]).

In Section 1 we briefly recall basic notions from the geometry of jet spaces. Section 2 contains main results on the Monge-Ampère equation (1) (including description of Hamiltonian, symplectic, and recursion operators, as well hierarchies of symmetries and conservation laws). In particular, we show that Equation (1) admits a symplectic structure of the form D_x (this is the only *local* operator that is responsible for the integrability of the equation and it corresponds to the symplectic structure described in [3, 18]). A nonlocal Hamiltonian structure D_x^{-1} corresponds to this operator. Other operators are quite complicated and are described in Sections 2.4.1, 2.4.2, 2.5.1, and 2.5.2.

All computations were done using `CDIFF`, a `REDUCE` package for computations in geometry of differential equations (see `http://gdeq.org`).

1. Theoretical background

1.1. **Jets and equations.** Recall that geometric approach to PDEs [1] assumes that an equation $\mathscr{E}$ together with all its prolongations (i.e., differential consequences) is a submanifold in the manifold $J^\infty(\pi)$ of infinite jets of some bundle $\pi\colon E \to M$, where M and E are smooth manifolds of dimensions n and $n+m$, respectively.

The first manifold is the one that contains independent variables, while the sections of π play the role of unknown functions (fields) in $\mathscr{E}$. If $U \subset M$ is a coordinate neighborhood such that $\pi|_U$ is trivial then we choose local coordinates $x^1, \dots, x^n$ in U and $u^1, \dots, u^m$ in the fiber of $\pi|_U$. Then the corresponding *adapted coordinates* u^j_σ, σ being a multi-index, in $J^\infty(\pi)$ are defined as follows. For a local section $f = (f^1, \dots, f^m)$ we set

$$f^*(u^j_\sigma) = \frac{\partial^{|\sigma|} f^j}{\partial x^\sigma}.$$

Functions on $J^\infty(\pi)$ may depend on x^i and *finite* number of u^j_σ only.

The vector fields

$$D_i = \frac{\partial}{\partial x^i} + \sum_{j,\sigma} u^j_{\sigma i} \frac{\partial}{\partial u^j_\sigma} \qquad i = 1, \dots, n$$

are called *total derivatives* and differential operators in total derivatives are called *$\mathscr{C}$-differential operators.*

If an equation is given by the system $F = 0$, where $F = (F^1, \dots, F^r)$ is a vector-function on $J^\infty(\pi)$, then its *infinite prolongation* $\mathscr{E}$ is given by

$$D_\sigma(F) = 0, \qquad |\sigma| \geq 0,$$

where $D_\sigma = D_{\sigma_1} \circ \dots \circ D_{\sigma_s}$ for $\sigma = \sigma_1 \dots \sigma_s$. Total derivatives can be restricted to $\mathscr{E}$ (we preserve the same notation for these restrictions) and generate the *Cartan distribution* $\mathscr{C}$. This distribution is integrable in a formal sense, i.e., $[X, Y] \in \mathscr{C}$ for any X, $Y \in \mathscr{C}$, and its n-dimensional integral manifolds are solutions of $\mathscr{E}$.

1.2. **Symmetries.** Denote by $\pi_\infty\colon \mathcal{E} \to M$ the natural projection. A π_∞-vertical vector field X on $\mathcal{E}$ is called a *symmetry* of $\mathcal{E}$ if it preserves the Cartan distribution, i.e., if $[X, \mathcal{C}] \subset \mathcal{C}$. Every symmetry of $\mathcal{E}$ is of the form

$$\mathcal{Э}_\varphi = \sum D_\sigma(\varphi^j) \frac{\partial}{\partial u^j_\sigma},$$

where summation is taken over *internal coordinates* on $\mathcal{E}$ and the vector-function $\varphi = (\varphi^1, \dots, \varphi^m)$ satisfies the equation

$$\ell_{\mathcal{E}}(\varphi) = 0.$$

Here $\ell_{\mathcal{E}}$ is the *linearization operator* of the vector function F restricted to $\mathcal{E}$. To be more precise, we take the functions F^α that define the equation $\mathcal{E}$ and construct the matrix $\mathcal{C}$-differential operator

$$\ell_F = \begin{pmatrix} \sum_\sigma \frac{\partial F^1}{\partial u^1_\sigma} D_\sigma & \dots & \sum_\sigma \frac{\partial F^1}{\partial u^m_\sigma} D_\sigma \\ \dots & \dots & \dots \\ \sum_\sigma \frac{\partial F^r}{\partial u^1_\sigma} D_\sigma & \dots & \sum_\sigma \frac{\partial F^r}{\partial u^m_\sigma} D_\sigma \end{pmatrix}.$$

Since thus defined operator is a $\mathcal{C}$-differential operator, it can be restricted to $\mathcal{E}$ and we set

$$\ell_{\mathcal{E}} = \ell_F|_{\mathcal{E}}.$$

The function φ is called the *generating function* (or *section*, or *characteristic*) of the corresponding symmetry $\mathcal{Э}_\varphi$ and we usually make no distinction between symmetries and their generating functions. The set of symmetries is a Lie algebra over $\mathbb{R}$ with respect to the commutator. We denote this algebra by $\operatorname{sym}(\mathcal{E})$. The bracket of vector fields induces a bracket of generating functions by

$$\mathcal{Э}_{\{\varphi_1,\varphi_2\}} = [\mathcal{Э}_{\varphi_1}, \mathcal{Э}_{\varphi_2}],$$

which is called the *Jacobi bracket* and is presented by

$$\{\varphi_1, \varphi_2\}^j = \sum_{\alpha,\sigma} \Big(D_\sigma(\varphi_1^\alpha) \frac{\partial \varphi_2^j}{\partial u^\alpha_\sigma} - D_\sigma(\varphi_2^\alpha) \frac{\partial \varphi_1^j}{\partial u^\alpha_\sigma} \Big)$$

in coordinates.

1.3. **Conservation laws.** Consider the space $\Lambda^1(\mathcal{E})$ of differential 1-forms on $\mathcal{E}$. It consists of finite sums

$$\omega = \sum_i A_i\, dx^i + \sum_{j,\sigma} B^\sigma_j\, du^j_\sigma,$$

A_i and B^σ_j being smooth functions on $\mathcal{E}$. The space $\Lambda^1(\mathcal{E})$ splits naturally into the direct sum

$$\Lambda^1(\mathcal{E}) = \Lambda^1_h(\mathcal{E}) \oplus \Lambda^1_v(\mathcal{E}), \tag{3}$$

where

$$\Lambda^1_h(\mathcal{E}) = \{\, \omega \in \Lambda^1(\mathcal{E}) \mid \omega = \sum A_i\, dx^i \,\}$$

is the subspace of *horizontal* forms, while $\Lambda^1_v(\mathcal{E})$ is generated by the differential forms $\omega^j_\sigma = du^j_\sigma - \sum_i u^j_{\sigma i}\, dx^i$ and is the subspace of *vertical* (or *Cartan*) forms.

Splitting (3) generates the splitting

$$\Lambda^s(\mathscr{E}) = \sum_{p+q=s} \Lambda_h^q(\mathscr{E}) \otimes \Lambda_v^p(\mathscr{E}),$$

where

$$\Lambda_h^q(\mathscr{E}) = \underbrace{\Lambda_h^1(\mathscr{E}) \wedge \dots \wedge \Lambda_h^1(\mathscr{E})}_{q \text{ times}}, \quad \Lambda_v^p(\mathscr{E}) = \underbrace{\Lambda_v^1(\mathscr{E}) \wedge \dots \wedge \Lambda_v^1(\mathscr{E})}_{p \text{ times}}.$$

Let us introduce the notation $\Lambda^{p,q}(\mathscr{E}) = \Lambda_h^q(\mathscr{E}) \otimes \Lambda_v^p(\mathscr{E})$. Consequently, the de Rham differential $d\colon \Lambda^s(\mathscr{E}) \to \Lambda^{s+1}(\mathscr{E})$ splits into the sum of the horizontal

$$d_h\colon \Lambda^{p,q}(\mathscr{E}) \to \Lambda^{p,q+1}(\mathscr{E})$$

and vertical

$$d_v\colon \Lambda^{p,q}(\mathscr{E}) \to \Lambda^{p+1,q}(\mathscr{E})$$

parts and one has

$$[d_h, d_v] = 0. \tag{4}$$

Due to (4), we have a bi-complex structure on $\Lambda^*(\mathscr{E})$ which is a particular case of Vinogradov's $\mathscr{C}$-spectral sequence [15, 16, 17]. Denote by $E_1^{p,q}(\mathscr{E})$ the cohomology of d_h at the term $\Lambda^{p,q}(\mathscr{E})$. Then d_v induces the differentials

$$\delta\colon E_1^{p,q}(\mathscr{E}) \to E_1^{p+1,q}(\mathscr{E}).$$

The group $E_1^{0,n-1}(\mathscr{E})$ plays a special role in the theory. Its elements are called *conservation laws* of $\mathscr{E}$, while the group itself is denoted by $\mathrm{Cl}(\mathscr{E})$. We also shall need the group $E_1^{1,n-1}$ whose elements are called *cosymmetries* and which is denoted by $\mathrm{cosym}(\mathscr{E})$.

To proceed, we shall need additional constructions. Let P and Q be the spaces of sections of vector bundles over $\mathscr{E}$. Let $\hat{P} = \mathrm{Hom}(P, \Lambda^n(\mathscr{E}))$ and similar for Q. Then for any $\mathscr{C}$-differential operator $\Delta\colon P \to Q$ its *formally adjoint* $\Delta^*\colon \hat{Q} \to \hat{P}$ is defined by the *Green formula*

$$\langle \Delta^*(\hat{q}), p\rangle - \langle \hat{q}, \Delta(p)\rangle = d_h\omega(\hat{q}, p),$$

where $\omega\colon \hat{Q} \times P \to \Lambda^{n-1}(\mathscr{E})$ is a map which is a $\mathscr{C}$-differential operator in both arguments and $\langle \cdot\,, \cdot\rangle$ denotes the natural pairing. If Δ is given by the matrix (Δ_{ij}), where $\Delta_{ij} = \sum_\sigma a_{ij}^\sigma D_\sigma$, then $\Delta^* = (\Delta_{ji}^*)$, where

$$\Delta_{ji}^* = \sum_\sigma (-1)^{|\sigma|} D_\sigma \circ a_{ji}^\sigma.$$

In what follows, we shall assume that equation at hand satisfies the following conditions:

(1) The differentials dF^j of the functions that define $\mathscr{E}$ are linear independent at all points of $\mathscr{E}$.
(2) If Δ is a $\mathscr{C}$-differential operator such that $\Delta \circ \ell_{\mathscr{E}} = 0$ then $\Delta = 0$.
(3) If Δ is a $\mathscr{C}$-differential operator such that $\Delta \circ \ell_{\mathscr{E}}^* = 0$ then $\Delta = 0$.

If an equation enjoys these conditions then the following statements are valid[1]:

[1]They follow from Vinogradov's 2-Line Theorem [17].

(1) The differential $\delta\colon \mathrm{Cl}(\mathscr{E}) \to \mathrm{cosym}(\mathscr{E})$ is monomorphic, i.e., $\delta(\omega) = 0$ if and only if $\omega = 0$.
(2) The group of cosymmetries coincides with the kernel of $\ell^*_{\mathscr{E}}$, i.e., $\psi \in \mathrm{cosym}(\mathscr{E})$ if and only if $\ell^*_{\mathscr{E}}(\psi) = 0$.

If $\omega \in \mathrm{Cl}(\mathscr{E})$ is a conservation law then the cosymmetry $\delta(\omega)$ is called its *generating function* (or *generating section*).

1.4. **Differential coverings.** Let $\mathscr{E}$ and $\tilde{\mathscr{E}}$ be two equations. A smooth map $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ is called a *morphism* if it takes the Cartan distribution on $\tilde{\mathscr{E}}$ to that on $\mathscr{E}$. A surjective morphism τ is said to be a *covering* if for any point $\theta \in \tilde{\mathscr{E}}$ the differential $d\tau|_\theta$ maps the Cartan plane $\mathscr{C}_\theta(\tilde{\mathscr{E}})$ to $\mathscr{C}_{\tau(\theta)}(\mathscr{E})$ isomorphically. Coordinates along the fibers of τ are called *nonlocal variables* in the covering under consideration. Let $\tau'\colon \tilde{\mathscr{E}}' \to \mathscr{E}$ be another covering. We say that it is *equivalent* to τ if there exists a morphism $f\colon \tilde{\mathscr{E}} \to \tilde{\mathscr{E}}'$ which is a diffeomorphism and such that $\tau = \tau' \circ f$.

If $D_1, \dots, D_n$ are total derivatives on $\mathscr{E}$ and $w^1, \dots, w^r, \dots$ are nonlocal variables then the covering structure is given by vector fields

$$\tilde{D}_i = D_i + X_i, \qquad i = 1, \dots, n, \tag{5}$$

where $X_i = \sum_\alpha X_i^\alpha \partial/\partial w^\alpha$ are τ-vertical fields that satisfy the condition

$$D_i(X_j) - D_j(X_i) + [X_i, X_j] = 0, \qquad 1 \le i < j \le n. \tag{6}$$

A covering is *Abelian* if the coefficients X_i^α do not depend on nonlocal variables. In this case, (6) amounts to

$$D_i(X_j) - D_j(X_i) = 0, \qquad 1 \le i < j \le n. \tag{7}$$

In the particular case of one-dimensional coverings, conditions (7) define a d_h-closed horizontal 1-form $\omega_\tau = \sum_i X_i\, dx^i$ on $\mathscr{E}$ and two coverings of this type are equivalent if and only if the corresponding forms are in the same cohomology class, i.e., $\omega_\tau - \omega_{\tau'} = d_h(g)$ for some function g. When n (the number of independent variables) equals two, this establishes a one-to-one correspondence between the group $\mathrm{Cl}(\mathscr{E})$ and the equivalence classes of one-dimensional Abelian coverings over $\mathscr{E}$.

If $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ is a covering then symmetries of $\tilde{\mathscr{E}}$ are called *nonlocal τ-symmetries* of $\mathscr{E}$. Note also that any $\mathscr{C}$-differential operator Δ on $\mathscr{E}$ can be lifted to a $\mathscr{C}$-differential operator $\tilde{\Delta}$ on $\tilde{\mathscr{E}}$. This is being done by changing total derivatives D_i in the local representation of Δ to $\tilde{D}_i$ using (5). In particular, the linearization operator $\ell_{\mathscr{E}}$ can be lifted in such a way and solutions of the equation

$$\tilde{\ell}_{\mathscr{E}}(\varphi) = 0$$

are called (nonlocal) *shadows of symmetries* in the covering τ. In a similar way, solutions of

$$\tilde{\ell}^*_{\mathscr{E}}(\psi) = 0$$

are (nonlocal) *shadows of cosymmetries* in the covering τ.

1.5. **The ℓ-covering.** Let $\mathscr{E}$ be an equation. Consider a new set of dependent variables $q = (q^1, \dots, q^m)$ (in many respects it is convenient to consider q as an odd variable), where m is the number of unknown functions in $\mathscr{E}$, and augment the initial equation with

$$\ell_{\mathscr{E}}(q) = 0. \tag{8}$$

The resulting system, consisting of $\mathscr{E}$ and equation (8), is called the *ℓ-covering* of $\mathscr{E}$. It is an analog of the tangent bundle for the equation $\mathscr{E}$. The ℓ-covering is important for the subsequent computations due to the following properties.

1.5.1. *Recursion operators for symmetries.* Consider a vector-function $\Phi = (\Phi^1, \dots, \Phi^m)$, where $\Phi^j = \sum_{\alpha,\sigma} \Phi^j_{\alpha,\sigma} q^\alpha_\sigma$, which is a symmetry shadow in the ℓ-covering. This means that it satisfies the equation

$$\tilde{\ell}_{\mathscr{E}}(\Phi) = 0, \tag{9}$$

where $\tilde{\ell}_{\mathscr{E}}$ is the linearization operator lifted to the ℓ-covering. Then it can be shown that the matrix $\mathscr{C}$-differential operator $\mathscr{R}_\Phi = (\sum_\sigma \Phi^j_{\alpha,\sigma} D_\sigma)$ takes symmetries of $\mathscr{E}$ to symmetries. In other words, $\mathscr{R}_\Phi$ is a recursion operator for symmetries.

A recursion operator $\mathscr{R}$ is called *hereditary* if

$$\{\mathscr{R}\varphi_1, \mathscr{R}\varphi_2\} - \mathscr{R}\big(\{\mathscr{R}\varphi_1, \varphi_2\} + \{\varphi_1, \mathscr{R}\varphi_2\} - \mathscr{R}\{\varphi_2, \varphi_2\}\big) = 0$$

Hereditary operators possess the following property important for integrability: let φ be a symmetry such that

$$\mathsf{Э}_\varphi(\mathscr{R}) - [\ell_\varphi, \mathscr{R}] = 0.$$

Then all symmetries $\mathscr{R}^i\varphi$ pair-wise commute, i.e., form a commutative hierarchy.

1.5.2. *Symplectic operators.* In a similar way, let us consider now a vector-function $\Psi = (\Psi^1, \dots, \Psi^r)$, where $\Psi^j = \sum_{\alpha,\sigma} \Psi^j_{\alpha,\sigma} q^\alpha_\sigma$ (recall that r is the number of the functions F^j that define $\mathscr{E}$), that satisfies the equation

$$\tilde{\ell}^*_{\mathscr{E}}(\Psi) = 0, \tag{10}$$

where $\tilde{\ell}^*_{\mathscr{E}}$ is the lift of the operator $\ell^*_{\mathscr{E}}$ to the ℓ-covering. Then the operator $\mathscr{S}_\Psi = (\sum_\sigma \Psi^j_{\alpha,\sigma} D_\sigma)$ takes symmetries of the equation $\mathscr{E}$ to its cosymmetries.

Let an operator $\mathscr{S}$ satisfy the condition

$$\mathscr{S}^* \circ \ell_{\mathscr{E}} = \ell^*_{\mathscr{E}} \circ \mathscr{S}. \tag{11}$$

Then $\mathscr{S}$ is identified with a *variational* 2*-form* $\Omega_{\mathscr{S}}$ on $\mathscr{E}$ whose values on symmetries is given by

$$\Omega_{\mathscr{S}}(\varphi_1, \varphi_2) = \langle \mathscr{S}\varphi_1, \varphi_2 \rangle.$$

This form can be considered as an element of the group $E_1^{2,n-1}(\mathscr{E})$ in the term E_1 of the $\mathscr{C}$-spectral sequence.

Let now ω_1, ω_2 be two conservation laws such that

$$\delta\omega_i = \mathscr{S}\varphi_i, \qquad i = 1, 2,$$

for some $\varphi_1,\ \varphi_2 \in \operatorname{sym} \mathscr{E}$. Then the bracket

$$\{\omega_1, \omega_2\}_{\mathscr{S}} = \Omega_{\mathscr{S}}(\varphi_1, \varphi_2)$$

is defined. This bracket is skew-symmetric by (11) and satisfies the Jacobi identity if

$$\delta\Omega_{\mathscr{S}} = 0. \tag{12}$$

Operators that enjoy properties (11) and (12) are called *symplectic.*

1.5.3. *Nonlocal covectors.* Solving equations (9) or (10) leads often to trivial results only. The reason is that symplectic and especially recursion operators are in many cases nonlocal, i.e., contain terms like D_x^{-1}. Such terms are incorporated into solution by introducing nonlocal variables that amounts to constructing appropriate coverings. One of the ways to construct the latter is based on the following fact (see [12]): *to any cosymmetry of the equation $\mathscr{E}$ there corresponds a conservation law on the ℓ-covering.* We call these conservation laws *nonlocal covectors.* Consequently, if $n = 2$ an Abelian covering corresponds to a cosymmetry. Numerous computations [7, 10, 11] show that nonlocal variables arising in such a way are sufficient to find necessary structures.

1.6. **The ℓ^*-covering.** Let again $\mathscr{E}$ be an equation. Consider a another new set of dependent variables $p = (p^1, \dots, p^r)$, where r is the number of functions F^j that determine the equation $\mathscr{E}$, and augment the initial equation with

$$\ell_{\mathscr{E}}^*(p) = 0. \tag{13}$$

The resulting system, consisting of $\mathscr{E}$ and equation (13), is called the ℓ^*-*covering* of $\mathscr{E}$. It is an analog of the cotangent bundle for the equation $\mathscr{E}$. The ℓ^*-covering is also important for the subsequent computations due to the following properties. Like the variable q in the ℓ-covering, it is convenient to consider the variable p to be odd.

1.6.1. *Hamiltonian operators.* Consider a vector-function $\Phi = (\Phi^1, \dots, \Phi^m)$, where $\Phi^j = \sum_{\alpha,\sigma} \Phi^j_{\alpha,\sigma} p^\alpha_\sigma$, and assume that it is a symmetry shadow in the ℓ^*-covering. This means that it satisfies the equation

$$\tilde{\ell}_{\mathscr{E}}(\Phi) = 0, \tag{14}$$

where $\tilde{\ell}_{\mathscr{E}}$ is the linearization operator lifted to the ℓ^*-covering. Then it can be shown that the the matrix $\mathscr{C}$-differential operator $\mathscr{H}_\Phi = (\sum_\sigma \Phi^j_{\alpha,\sigma} D_\sigma)$ takes cosymmetries of $\mathscr{E}$ to symmetries.

Solutions to (14) of special type are identified with *variational bivectors* $\Lambda_{\mathscr{H}}$ on $\mathscr{E}$. These solutions must satisfy the condition

$$\ell_{\mathscr{E}} \circ \mathscr{H} = \mathscr{H}^* \circ \ell_{\mathscr{E}}^*.$$

In this case, the operation

$$\{\omega_1, \omega_2\}_{\mathscr{H}} = \langle \mathscr{H}(\delta\omega_1), \delta\omega_2 \rangle, \qquad \omega_1, \omega_2 \in \mathrm{Cl}(\mathscr{E}),$$

defines a skew-symmetric bracket on the space of conservation laws. This bracket satisfies the Jacobi identity if and only if $[\![\Lambda_{\mathscr{H}}, \Lambda_{\mathscr{H}}]\!] = 0$, where $[\![\cdot,\cdot]\!]$

is the *variational Schouten bracket* (see [10, 12]) on the space of variational multi-vectors. In this case, one has

$$\mathscr{H}\delta(\{\omega_1,\omega_2\}_{\mathscr{H}}) = \{\mathscr{H}\delta\omega_1,\mathscr{H}\delta\omega_2\}.$$

1.6.2. *Recursion operators for cosymmetries.* Finally, let us now consider a vector-function $\Psi = (\Psi^1,\dots,\Psi^r)$, where $\Psi^j = \sum_{\alpha,\sigma}\Psi^j_{\alpha,\sigma}p^\alpha_\sigma$, and assume that it is a cosymmetry shadow in the ℓ^*-covering. This means that it satisfies the equation

$$\tilde{\ell}^*_{\mathscr{E}}(\Psi) = 0, \tag{15}$$

where $\tilde{\ell}_{\mathscr{E}}$ is the linearization operator lifted to the ℓ^*-covering. Then it can be shown that the the matrix $\mathscr{C}$-differential operator $\hat{\mathscr{R}}_\Psi = (\sum_\sigma \Psi^j_{\alpha,\sigma}D_\sigma)$ takes cosymmetries of $\mathscr{E}$ to cosymmetries. In other words, $\hat{\mathscr{R}}$ is a recursion operator for cosymmetries of $\mathscr{E}$.

1.6.3. *Nonlocal vectors.* Similar to Section 1.5, equations (14) and (15) lead often to trivial results only. The reason is the same: Hamiltonian and recursion operators are in many cases nonlocal. Such terms are incorporated into solution by introducing nonlocal variables that amounts to constructing appropriate coverings. One of the ways to construct the latter is based on the following fact: *to any symmetry of the equation $\mathscr{E}$ there corresponds a conservation law on the ℓ^*-covering.* We call these conservation laws *nonlocal vectors.* Consequently, if $n = 2$ an Abelian covering corresponds to a symmetry. As computations [7, 10, 11] show, nonlocal variables arising in such a way are sufficient to find necessary structures.

1.7. **General computational scheme.** In all the computations we did to analyze particular equations (Equation (1) included) we adhered to the following scheme:

- Extension of the initial equation with a minimal set of nonlocal variables (usually associated with conservation laws) to ensure existence of nontrivial solutions to the main equations defining integrable structures.
- Computation of a minimal set of (local and nonlocal) symmetries and cosymmetries necessary to (a) hierarchy generation and (b) construction of nonlocal vectors and covectors.
- Extension of the ℓ-covering and and construction of symplectic structures and recursion operators for symmetries.
- Extension of the ℓ^*-covering and and construction of Hamiltonian structures and recursion operators for cosymmetries.

2. Main results

For internal coordinates on $\mathscr{E}$ we choose the functions

$$u_{k,i} = \frac{\partial^{k+i}u}{\partial x^k\partial y^i}, \qquad i = 0,1,2, \quad k = 0,1,\dots, \tag{16}$$

and then the total derivatives on $\mathscr{E}$ acquire the form

$$D_x = \frac{\partial}{\partial x} + \sum_{k\ge 0}(u_{k+1,0}\frac{\partial}{\partial u_{k,0}} + u_{k+1,1}\frac{\partial}{\partial u_{k,1}} + u_{k+1,2}\frac{\partial}{\partial u_{k,2}}), \tag{17}$$

$$D_y = \frac{\partial}{\partial y} + \sum_{k\geq 0}(u_{k,1}\frac{\partial}{\partial u_{k,0}} + u_{k,2}\frac{\partial}{\partial u_{k,1}} + D_x^k(u_{2,1}^2 - u_{3,0}u_{1,2})\frac{\partial}{\partial u_{k,2}}).$$

Equation (1) is homogeneous with respect to the following weights

$$|u| = 0, \quad |x| = -1, \quad |y| = -4.$$

2.1. **Conservation laws and Abelian coverings.** In the sequel, we shall need nonlocal variables that will be denoted by $Q_{i,j}$. The second subscript here indicates the weight of the variable, while first the one corresponds to the *level of nonlocality*. By the latter we mean the following. The variables of zero level are determined by local functions on $\mathscr{E}$:

$$\frac{\partial Q_{0,7}}{\partial x} = -u_{0,1}u_{4,0} + u_{0,2}, \quad \frac{\partial Q_{0,7}}{\partial y} = -u_{0,1}u_{3,1} - u_{0,2}u_{3,0} + u_{1,1}u_{2,1},$$
$$\frac{\partial Q_{0,9}}{\partial x} = 2u_{0,2}u_{2,0} + u_{1,1}^2 - u_{2,0}^2u_{2,1}, \quad \frac{\partial Q_{0,9}}{\partial y} = 2u_{0,2}u_{1,1} - u_{1,2}u_{2,0}^2,$$

and

$$\frac{\partial Q_{0,12}}{\partial x} = u_{1,1}(u_{0,2} - u_{2,0}u_{2,1}), \quad \frac{\partial Q_{0,12}}{\partial y} = \frac{1}{2}(u_{0,2}^2 - 2, u_{1,1}, u_{1,2}u_{2,0}).$$

The variables of level 1 are determined by local functions and by variables of zero level:

$$\frac{\partial Q_{1,3}}{\partial x} = 2u_{0,1} - u_{2,0}^2, \quad \frac{\partial Q_{1,3}}{\partial y} = 2(Q_{0,7} + u_{0,1}u_{3,0} - u_{1,1}u_{2,0}),$$
$$\frac{\partial Q_{1,6}}{\partial x} = Q_{0,7} + u_{0,1}u_{3,0}, \quad \frac{\partial Q_{1,6}}{\partial y} = \frac{1}{2}u_{1,1}^2,$$

and

$$\begin{aligned}\frac{\partial Q_{1,8}}{\partial x} &= Q_{0,9} - 2u_{0,1}u_{1,0}u_{4,0} - 4u_{0,1}u_{20}u_{30} + 2u_{0,2}u_{1,0},\\ \frac{\partial Q_{1,8}}{\partial y} &= 4Q_{0,12} - 2u_{0,1}u_{1,0}u_{3,1} - 2u_{0,1}u_{1,1}u_{3,0} - 2u_{0,1}u_{2,0}u_{2,1}\\ &\quad - 2u_{0,2}u_{1,0}u_{3,0} - u_{0,2}u_{2,0}^2 + 2u_{1,0}u_{1,1}u_{2,1} + 2u_{1,1}^2u_{2,0}.\end{aligned}$$

There exist deeper level nonlocalities, such as

$$\begin{aligned}\frac{\partial Q_{2,5}}{\partial x} &= -18Q_{1,6} - 2u_{0,1}u_{2,0} + 4u_{1,0}u_{1,1} + u_{2,0}^3,\\ \frac{\partial Q_{2,5}}{\partial y} &= -3Q_{0,9} - 2u_{0,1}u_{1,1} + 4u_{0,2}u_{1,0}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial Q_{2,7}}{\partial x} &= -40Q_{0,7}u_{1,0} - 10Q_{1,8} - 10u_{0,1}^2 - 60u_{0,1}u_{1,0}u_{3,0}\\ &\quad + u_{1,0}u_{2,0}^2u_{3,0} - \frac{1}{2}u_{20}^4,\\ \frac{\partial Q_{2,7}}{\partial y} &= -40Q_{0,7}u_{01} - 10Q_{1,11} - 30u_{0,1}^2u_{3,0} + 20u_{0,1}u_{1,1}u_{2,0}\\ &\quad + 10u_{0,1}u_{2,0}^2u_{3,0} - 30u_{1,0}u_{1,1}^2 + u_{1,0}u_{2,0}^2u_{2,1} - 3u_{1,1}u_{2,0}^3,\end{aligned}$$

as well as

$$\frac{\partial Q_{3,4}}{\partial x} = \frac{1}{3}Q_{2,5} + Q_{1,3}u_{2,0} - \frac{4}{3}u_{0,1}u_{1,0},$$
$$\frac{\partial Q_{3,4}}{\partial y} = 2Q_{0,7}u_{1,0} + Q_{1,3}u_{1,1} - Q_{1,8} - 2u_{0,1}^2 - u_{0,1}u_{2,0}^2.$$

Remark 1. The zero level nonlocal variables are associated to conservation laws of equation (1). For example, to $Q_{0,7}$ there corresponds the conservation law

$$\omega_{0,7} = (-u_{0,1}u_{4,0} + u_{0,2})\,dx + (-u_{0,1}u_{3,1} - u_{0,2}u_{3,0} + u_{1,1}u_{2,1})\,dy.$$

The first level nonlocal variables are associated to conservation laws of the Abelian coverings determined by the zero level variables, etc.

Remark 2. Of course, the list of nonlocal variables above is not exhaustive at all. We described only those ones that are used to construct the necessary nonlocal symmetries (§ 2.2) and cosymmetries (§ 2.3); see also § 1.7. New nonlocalities do arise under the actions of recursion operators, but there is no need to describe them explicitly here.

2.2. **Symmetries.** This direct computation is needed by the two reasons: to construct nonlocal vectors (see Subsection 1.6.3) and to use the obtained symmetries as 'seeds' for the hierarchies.

The linearization of equation (1) has the form

$$D_y^3(\varphi) - 2u_{2,1}D_x^2D_y(\varphi) + u_{1,2}D_x^3(\varphi) + u_{3,0}D_xD_y^2(\varphi) = 0, \tag{18}$$

where the total derivatives D_x and D_y are given by (17).

Solving (18), we get the following solutions[2].

2.2.1. *Symmetries of degree* 0:

$$\varphi_0^0 = 1, \quad \varphi_1^0 = u_{1,0}, \quad \varphi_4^0 = u_{0,1}, \quad \varphi_5^0 = Q_{2,5} + 8u_{0,1}u_{1,0},$$
$$\varphi_8^0 = -2Q_{0,7}u_{1,0} + Q_{1,8} - 2u_{0,1}^2 + u_{0,1}u_{2,0}^2.$$

2.2.2. *Symmetries of degree* 1:

$$\varphi_{-4}^1 = y, \quad \varphi_{-1}^1 = x,$$
$$\varphi_{0,1}^1 = xu_{1,0} - 4u, \qquad \varphi_{0,2}^1 = yu_{0,1} + u,$$
$$\varphi_3^1 = 4xu_{0,1} - Q_{1,3},$$
$$\varphi_4^1 = x(Q_{2,5} + 8u_{0,1}u_{1,0}) - 3(8Q_{3,4} - 3Q_{1,3}u_{1,0} + 16uu_{0,1}).$$

2.2.3. *Symmetries of degree* 2:

$$\varphi_{-4}^2 = y, \quad \varphi_{-8}^2 = y^2, \quad \varphi_{-5}^2 = xy, \quad \varphi_{-2}^2 = x^2,$$
$$\varphi_{-1}^2 = x^2u_{1,0} + 4xyu_{0,1} - 4xu - yQ_{1,3},$$
$$\varphi_2^2 = 2x^2u_{0,1} - xQ_{1,3} - u_{1,0}^2.$$

[2]In the notation for symmetries, the superscript indicates the polynomial order of a symmetry with respect to x and y, the first subscript equals the weight, while the second one, if any, is the number of a symmetry in the set given by particular weight and order.

2.2.4. *Symmetries of degree* 3:

$$\varphi^3_{-3} = x^3 - 2yu_{1,0},$$
$$\varphi^3_1 = 12x^3u_{0,1} - 9x^2Q_{1,3} - 18xu^2_{1,0} - 2y(Q_{2,5} + 8u_{0,1}u_{1,0}) + 24uu_{1,0}.$$

2.2.5. *Symmetries of degree* 4: We shall also need one symmetry of order 4, which is of the form

$$\varphi^4_{-4} = x^4 - 8xyu_{1,0} - 8y^2u_{0,1} + 16yu.$$

2.3. **Cosymmetries.** The reasons to compute cosymmetries explicitly are similar to those indicated in Subsection 2.2.

To find cosymmetries, we are to solve the equation adjoint to (18), i.e.,

$$D^3_y(\psi) - 2D^2_xD_y(u_{2,1}\psi) + D^3_x(u_{1,2}\psi) + D_xD^2_y(u_{3,0}\psi) = 0. \tag{19}$$

Using the notation similar to the one from Section 2.2, let us some computational results needed below.

2.3.1. *Cosymmetries of degree* 0:

$$\psi^0_0 = 1, \quad \psi^0_2 = u_{2,0}, \quad \psi^0_5 = u_{1,1},$$
$$\psi^0_6 = -18Q_{1,6} + 6u_{0,1}u_{2,0} + 12u_{1,0}u_{1,1} + u^3_{2,0},$$
$$\psi^0_9 = 2Q_{0,7}u_{2,0} - Q_{0,9} + 4u_{0,1}u_{1,1} + 2u_{0,1}u_{2,0}u_{3,0} - u_{1,1}u^2_{2,0}.$$

2.3.2. *Cosymmetries of degree* 1:

$$\psi^1_{-4} = y, \quad \psi^1_{-1} = x, \quad \psi^1_{1,1} = xu_{2,0} - 3u_{1,0},$$
$$\psi^1_{1,2} = yu_{1,1} + u_{1,0}, \quad \psi^1_4 = 4xu_{1,1} + 2u_{0,1} + u^2_{2,0}.$$

2.3.3. *Cosymmetries of degree* 2:

$$\psi^2_{-2} = 3x^2 - 2yu_{2,0},$$
$$\psi^2_0 = x^2u_{2,0} + 4xyu_{1,1} - 2xu_{1,0} + y(2u_{0,1} + u^2_{2,0}) - 4u,$$
$$\psi^2_3 = 2x^2u_{1,1} + x(2u_{0,1} + u^2_{2,0}) - Q_{1,3} - 2u_{1,0}u_{2,0}.$$

2.3.4. *Cosymmetries of degree* 3:

$$\psi^3_{-3} = 2u_{1,0}y - 2u_{1,1}y^2 - 2u_{2,0}xy + x^3,$$
$$\psi^3_2 = x^3u_{1,1} + x^2\left(\frac{3}{2}u_{0,1} + \frac{3}{4}u^2_{2,0}\right) - x\left(\frac{3}{2}Q_{1,3} + 3u_{1,0}u_{2,0}\right)$$
$$+y\left(3Q_{1,6} - u_{0,1}u_{2,0} - 2u_{1,0}u_{1,1} - \frac{1}{6}u^3_{2,0}\right) + (2uu_{2,0} + \frac{1}{2}u^2_{1,0}).$$

2.4. **The ℓ-covering.** This covering is determined by the system of equations

$$u_{yyy} - u^2_{xxy} + u_{xxx}u_{xyy} = 0,$$
$$q_{yyy} - 2u_{xxy}q_{xxy} + u_{xyy}q_{xxx} + u_{xxx}q_{xyy} = 0,$$

where q is an odd variable along the fiber of the covering.

Internal coordinates on the space of the ℓ-covering are functions (16) together with

$$q_{k,i} = \frac{\partial^{k+i}q}{\partial x^k\partial y^i}, \qquad i = 0, 1, 2, \quad k = 0, 1, \dots, \tag{20}$$

while the total derivatives in these coordinates take the form

$$\begin{aligned} D_x &= \frac{\partial}{\partial x} + \sum_{k\geq 0}\Big(u_{k+1,0}\frac{\partial}{\partial u_{k,0}} + u_{k+1,1}\frac{\partial}{\partial u_{k,1}} + u_{k+1,2}\frac{\partial}{\partial u_{k,2}} \\ &\quad + q_{k+1,0}\frac{\partial}{\partial q_{k,0}} + q_{k+1,1}\frac{\partial}{\partial q_{k,1}} + q_{k+1,2}\frac{\partial}{\partial q_{k,2}}\Big), \\ D_y &= \frac{\partial}{\partial y} + \sum_{k\geq 0}\Big(u_{k,1}\frac{\partial}{\partial u_{k,0}} + u_{k,2}\frac{\partial}{\partial u_{k,1}} + D_x^k(u_{2,1}^2 - u_{3,0}u_{1,2})\frac{\partial}{\partial u_{k,2}} \\ &\quad + q_{k,1}\frac{\partial}{\partial q_{k,0}} + q_{k,2}\frac{\partial}{\partial q_{k,1}} + D_x^k(2u_{2,1}q_{2,1} - u_{1,2}q_{3,0} - u_{3,0}q_{1,2})\frac{\partial}{\partial q_{k,2}}\Big). \end{aligned}$$

By the general theory [12], to any cosymmetry ψ of the initial equation there corresponds a conservation law on the ℓ-covering (a nonlocal form). Denote by $\mathbf{\Omega}_\psi$ the corresponding nonlocal variable. In the case of Equation (1), this variable is defined by the relations

$$\begin{aligned} \frac{\partial \mathbf{\Omega}_\psi}{\partial x} &= \psi q_{0,2} + a_{0,1}q_{0,1} + a_{0,0}q, \\ \frac{\partial \mathbf{\Omega}_\psi}{\partial y} &= b_{0,2}q_{0,2} + b_{1,1}q_{1,1} + b_{2,0}q_{2,0} + b_{0,1}q_{0,1} + b_{1,0}q_{1,0} + b_{0,0}q, \end{aligned} \tag{21}$$

where

$$\begin{gathered} b_{0,2} = -u_{3,0}\psi, \quad b_{1,1} = 2u_{2,1}\psi, \quad b_{2,0} = -u_{1,2}\psi, \\ b_{0,1} = -D_x(b_{1,1}), \quad b_{1,0} = -D_x(b_{2,0}), \\ b_{0,0} = -D_x(b_{1,0}) \end{gathered} \tag{22}$$

and

$$a_{0,1} = D_x(b_{0,2}) - D_y(\psi), \quad a_{0,0} = D_x(b_{0,1}) - D_y(a_{0,1}). \tag{23}$$

Below we use the notation $\mathbf{\Omega}^k_{i,j} = \mathbf{\Omega}_{\psi^k_{i,j}}$

2.4.1. *Recursion operators for symmetries.* To find recursion operators for symmetries of Equation (1), we solve the equation

$$\tilde{\ell}_{\mathscr{E}}(\Phi) = 0,$$

where $\tilde{\ell}_{\mathscr{E}}$ is the linearization operator (18) in the ℓ-covering extended by nonlocal forms and Φ is a function on this extension.

The simplest nontrivial solution is of the form

$$\begin{aligned} \Phi_1 &= \mathbf{\Omega}^3_{-3} - x\mathbf{\Omega}^2_{-2} + 4y\mathbf{\Omega}^1_{1,2} + 2y\mathbf{\Omega}^1_{1,1} + 3x^2\mathbf{\Omega}^1_{-1} \\ &\quad - 2u_{1,0}\mathbf{\Omega}^1_{-4} - 2y^2\mathbf{\Omega}^0_5 - 2xy\mathbf{\Omega}^0_2 + (2u_{1,0}y - x^3)\mathbf{\Omega}^0_0. \end{aligned} \tag{24}$$

Using the first of Equations (21), we put into correspondence to every nonlocal form $\mathbf{\Omega}_\psi$ the operator

$$\mathscr{D}_\psi = D_x^{-1}\circ(\psi D_y^2 + a_{0,1}D_y + a_{0,0}) \tag{25}$$

where the coefficients are determined using relations (22) and (23), i.e.,

$$\begin{aligned} a_{0,1} &= -D_x(u_{3,0}\psi) - D_y(\psi), \\ a_{0,0} &= -2D_x^2(u_{2,1}\psi) + D_xD_y(u_{3,0}\psi) + D_y^2(\psi). \end{aligned}$$

Then the recursion operator

$$\begin{aligned}\mathscr{R}_1 &= \mathscr{D}_{\psi^3_{-3}} - x\mathscr{D}_{\psi^2_{-2}} + 4y\mathscr{D}_{\psi^1_{1,2}} + 2y\mathscr{D}_{\psi^1_{1,1}} + 3x^2\mathscr{D}_{\psi^1_{-1}} \\ &\quad - 2u_{1,0}\mathscr{D}_{\psi^1_{-4}} - 2y^2\mathscr{D}_{\psi^0_5} - 2xy\mathscr{D}_{\psi^0_2} + (2u_{1,0}y - x^3)\mathscr{D}_{\psi^0_0} \end{aligned} \tag{26}$$

corresponds to solution (24).

Remark 3. Since the variables x and y in Equation (1) "enjoy equal rights", one can put into correspondence to the nonlocal form $\mathbf{\Omega}_\psi$, using the second equality in (21), the operator

$$\mathscr{D}'_\psi = D_x^{-1} \circ (b_{0,2}D_y^2 + b_{1,1}D_xD_y + b_{2,0}D_x^2 + b_{0,1}D_y + b_{1,0}D_x + b_{0,0})$$

and construct a recursion operator $\mathscr{R}'_1$ similar to operator (26).

The action of operators $\mathscr{R}_1$ and $\mathscr{R}'_1$ on symmetries of Equation (1) is the same.

2.4.2. *Symplectic structures.* To find symplectic structures, we solve the equation

$$\tilde{\ell}^*_{\mathscr{E}}(\Psi) = 0,$$

where, similar to Section 2.4.1, $\tilde{\ell}^*_{\mathscr{E}}$ is the operator (19) on the ℓ-covering extended by the nonlocal forms, while Ψ is a function on this extension.

Below we present the first two solutions of this equation. The simplest one is

$$\Psi_1 = \mathbf{\Omega}_{1,0},$$

and the corresponding symplectic structure

$$\mathscr{S}_1 = D_x. \tag{27}$$

The next solution is nonlocal:

$$\Psi_2 = \mathbf{\Omega}^2_{-1} - 6x\mathbf{\Omega}^1_{-1} + 2u_{2,0}\mathbf{\Omega}^1_{-4} + 2y\mathbf{\Omega}^0_2 - (2u_{2,0}y - 3x^2)\mathbf{\Omega}^0_0.$$

The corresponding symplectic operator $\mathscr{S}_2\colon \operatorname{sym}\mathscr{E} \to \operatorname{cosym}\mathscr{E}$ is

$$\mathscr{S}_2 = \mathscr{D}_{\psi^2_{-1}} - 6x\mathscr{D}_{\psi^1_{-1}} + 2u_{2,0}\mathscr{D}_{\psi^1_{-4}} + 2y\mathscr{D}_{\psi^0_2} - (2u_{2,0}y - 3x^2)\mathscr{D}_{\psi^0_0},$$

where the operators $\mathscr{D}_\psi$ are defined by Equation (25).

2.5. **The ℓ^*-covering.** As it was indicated above, this covering is obtained by adding to the initial equation another one, which is adjoint to the linearization. In other words, this covering is described by the equations

$$\begin{aligned} &u_{yyy} - u_{xxy}^2 + u_{xxx}u_{xyy} = 0, \\ &u_{xxyy}p_{xx} - 2u_{xxxy}p_{xy} + u_{xxxx}p_{yy} \\ &\qquad + u_{xyy}p_{xxx} - 2u_{xxy}p_{xxy} + u_{xxx}p_{xyy} + p_{yyy} = 0, \end{aligned}$$

where p is a new odd variable.

Internal coordinates in the space of the ℓ^*-covering are functions (16) together with the functions

$$p_{k,i} = \frac{\partial^{k+i}p}{\partial x^k \partial y^i}, \qquad i = 0, 1, 2, \quad k = 0, 1, \dots,$$

while the total derivatives in these coordinates are of the form

$$D_x = \frac{\partial}{\partial x} + \sum_{k\ge 0}\Big(u_{k+1,0}\frac{\partial}{\partial u_{k,0}} + u_{k+1,1}\frac{\partial}{\partial u_{k,1}} + u_{k+1,2}\frac{\partial}{\partial u_{k,2}}$$

$$+ p_{k+1,0}\frac{\partial}{\partial p_{k,0}} + p_{k+1,1}\frac{\partial}{\partial p_{k,1}} + p_{k+1,2}\frac{\partial}{\partial p_{k,2}}\Big),$$

$$D_y = \frac{\partial}{\partial y} + \sum_{k\geq 0}\Big(u_{k,1}\frac{\partial}{\partial u_{k,0}} + u_{k,2}\frac{\partial}{\partial u_{k,1}} + D_x^k(u_{2,1}^2 - u_{3,0}u_{1,2})\frac{\partial}{\partial u_{k,2}}$$
$$+ p_{k,1}\frac{\partial}{\partial p_{k,0}} + p_{k,2}\frac{\partial}{\partial p_{k,1}} - D_x^k(u_{2,2}p_{2,0} - 2u_{3,1}p_{1,1} + u_{4,0}p_{0,2}$$
$$+ u_{1,2}p_{3,0} - 2u_{2,1}p_{2,1} + u_{3,0}p_{1,2})\frac{\partial}{\partial p_{k,2}}\Big).$$

Let φ be a symmetry of Equation (1). Then (see [12]) a conservation law on the ℓ^*-covering corresponds to this symmetry and, consequently a nonlocal variable, which we denote by $\mathbf{\Pi}_\varphi$ and call a *nonlocal vector*. For Equation (1), the correspondence $\varphi \mapsto \mathbf{\Pi}_\varphi$ is given by the relations

$$\begin{aligned}\frac{\partial \mathbf{\Pi}_\varphi}{\partial x} &= \varphi p_{0,2} + a_{0,1}p_{0,1} + a_{0,0}p,\\ \frac{\partial \mathbf{\Pi}_\varphi}{\partial y} &= b_{0,2}p_{0,2} + b_{1,1}p_{1,1} + b_{2,0}p_{2,0} + b_{0,1}p_{0,1} + b_{1,0}p_{1,0} + b_{0,0}p,\end{aligned} \tag{28}$$

where

$$\begin{gathered} b_{0,2} = -u_{3,0}\varphi, \quad b_{1,1} = 2u_{2,1}\varphi, \quad b_{2,0} = -u_{1,2}\varphi,\\ b_{0,1} = -D_x(b_{1,1}) + 2u_{3,1}\varphi, \quad b_{1,0} = -D_x(b_{2,0}) - u_{2,2}\varphi,\\ b_{0,0} = -D_x(b_{1,0})\end{gathered} \tag{29}$$

and

$$a_{0,1} = D_x(b_{0,2}) - D_y(\varphi) + u_{4,0}\varphi, \quad a_{0,0} = D_x(b_{0,1}) - D_y(a_{0,1}).$$

We use the notation $\mathbf{\Pi}_{i,j}^k = \mathbf{\Pi}_{\varphi_{i,j}^k}$ below.

To describe the subsequent results, let us put into correspondence to a nonlocal vector $\mathbf{\Pi}_\varphi$ the operator

$$\mathscr{D}_\varphi = D_x^{-1} \circ (\varphi D_y^2 + a_{0,1}D_y + a_{0,0}), \tag{30}$$

see the first of Equations (28). Here, due to relations (29) and (30), the coefficients $a_{0,0}$ and $a_{0,1}$ are of the form

$$\begin{gathered} a_{0,0} = -2u_{2,1}D_x^2(\varphi) + u_{3,0}D_xD_y(\varphi) + D_y^2(\varphi) - u_{3,1}D_x(\varphi),\\ a_{0,1} = -u_{3,0}D_x(\varphi) - D_y(\varphi).\end{gathered}$$

2.5.1. *Hamiltonian structures.* Similar to Sections 2.4.1 and 2.4.2, Hamiltonian structures are solutions of the equation

$$\tilde{\ell}_{\mathscr{E}}(\Phi) = 0, \tag{31}$$

where the operator $\tilde{\ell}_{\mathscr{E}}$ is the linearization lifted to the $\tilde{\ell}^*$-covering extended by nonlocal vectors.

The simplest solution of Equation (31) is

$$\Phi_0 = \mathbf{\Pi}_{-8}^2 - 2y\mathbf{\Pi}_{-4}^1 + y^2\mathbf{\Pi}_0^0,$$

to which the operator $\mathscr{H}_0\colon \operatorname{cosym}\mathscr{E} \to \operatorname{sym}\mathscr{E}$

$$\mathscr{H}_0 = \mathscr{D}_{\varphi_{-8}^2} - 2y\mathscr{D}_{\varphi_{-4}^1} + y^2\mathscr{D}_{\varphi_0^0}$$

corresponds. The next solution is much more complicated and has the form

$$\begin{aligned}\Phi_1 &= \mathbf{\Pi}^4_{-4} - 4x\mathbf{\Pi}^3_{-3} + 6x^2\mathbf{\Pi}^2_{-2} - 8u_{1,0}\mathbf{\Pi}^2_{-5} - 8u_{0,1}\mathbf{\Pi}^2_{-8} + 16y\mathbf{\Pi}^1_{0,2}\\ &+ (8xu_{1,0} + 16yu_{0,1} - 16u)\mathbf{\Pi}^1_{-4} + 8y\mathbf{\Pi}^1_{0,1} - (4x^3 - 8yu_{1,0})\mathbf{\Pi}^1_{-1}\\ &- 8y^2\mathbf{\Pi}^0_4 - 8xy\mathbf{\Pi}^0_1 + (x^4 - 8xyu_{1,0} - 8y^2u_{0,1} + 16yu)\mathbf{\Pi}^0_0\end{aligned}$$

and the operator

$$\begin{aligned}\mathscr{H}_1 &= \mathscr{D}_{\varphi^4_{-4}} - 4x\mathscr{D}_{\varphi^3_{-3}} + 6x^2\mathscr{D}_{\varphi^2_{-2}} - 8u_{1,0}\mathscr{D}_{\varphi^2_{-5}} - 8u_{0,1}\mathscr{D}^2_{\varphi_{-8}} + 16y\mathscr{D}_{\varphi^1_{0,2}}\\ &+ (8xu_{1,0} + 16yu_{0,1} - 16u)\mathscr{D}_{\varphi^1_{-4}} + 8y\mathscr{D}_{\varphi^1_{0,1}} - (4x^3 - 8yu_{1,0})\mathscr{D}_{\varphi^1_{-1}}\\ &- 8y^2\mathscr{D}_{\varphi^0_4} - 8xy\mathscr{D}_{\varphi^0_1} + (x^4 - 8xyu_{1,0} - 8y^2u_{0,1} + 16yu)\mathscr{D}_{\varphi^0_0}.\end{aligned}$$

corresponds to this solution. Here and below the operators $\mathscr{D}_\varphi$ are defined by Equality (30).

Remark 4. The form of the above presented solutions is determined by the choice of nonlocal vectors in the ℓ^*-covering. They, in turn, are due to a basis in the space of symmetries. Actually, Equation (31) has a simpler solution $\Phi_0' = \mathbf{\Pi}_1'$, where the nonlocal variable $\mathbf{\Pi}_1'$ is defined by the system

$$\begin{aligned}\frac{\partial \mathbf{\Pi}_1'}{\partial x} &= p, & \frac{\partial \mathbf{\Pi}_1'}{\partial y} &= \mathbf{\Pi}_2',\\ \frac{\partial \mathbf{\Pi}_2'}{\partial x} &= p_{0,1}, & \frac{\partial \mathbf{\Pi}_2'}{\partial y} &= \mathbf{\Pi}_3',\\ \frac{\partial \mathbf{\Pi}_3}{\partial x} &= p_{0,2}, & \frac{\partial \mathbf{\Pi}_3'}{\partial y} &= -u_{3,0}p_{0,2} + 2u_{2,1}p_{1,1} - u_{1,2}p_{2,0}.\end{aligned}$$

To this solution there corresponds the Hamiltonian operator

$$\mathscr{H}_0' = D_x^{-1},$$

which is the inverse to the symplectic operator (27). This operator corresponds to the Hamiltonian operator J_0 from [9]. The operator $\mathscr{H}_1$ that explicitly depends on x and y seems to be new.

Remark 5. There is a relation

$$\mathscr{H}_1 = \mathscr{R}_1 \circ \mathscr{H}_0,$$

between the two Hamiltonian structures, where $\mathscr{R}_1$ is the recursion operator given by Equality (26).

2.5.2. *Recursion operators for cosymmetries.* To conclude our study, we consider finally the equation

$$\tilde{\ell}^*_{\mathscr{E}}(\Psi) = 0$$

on the ℓ^*-covering extended by nonlocal vectors. Here is its simplest nontrivial solution

$$\begin{aligned}\Psi_0 &= \mathbf{\Pi}^3_{-3} - 3x\mathbf{\Pi}^2_{-2} + 2u_{2,0}\mathbf{\Pi}^2_{-5} + 2u_{1,1}\mathbf{\Pi}^2_{-8}\\ &+ 2(u_{1,0} + 2u_{1,1}y - u_{2,0}x)\mathbf{\Pi}^1_{-4} - (2u_{2,0}y - 3x^2)\mathbf{\Pi}^1_{-1} + 2y\mathbf{\Pi}^0_1\\ &- (2u_{1,0}y - 2u_{1,1}y^2 - 2u_{2,0}xy + x^3)\mathbf{\Pi}^0_0.\end{aligned}$$

To this solution there corresponds the recursion operator $\hat{R}_0\colon \operatorname{cosym}\mathscr{E} \to \operatorname{cosym}\mathscr{E}$ of the form

$$\begin{aligned}\hat{R}_0 &= \mathscr{D}_{\varphi^3_{-3}} - 3x\mathscr{D}_{\varphi^2_{-2}} + 2u_{2,0}\mathscr{D}^2_{\varphi_{-5}} + 2u_{1,1}\mathscr{D}_{\varphi^2_{-8}}\\ &\quad + 2(u_{1,0} - 2u_{1,1}y - u_{2,0}x)\mathscr{D}_{\varphi^1_{-4}} - (2u_{2,0}y - 3x^2)\mathscr{D}_{\varphi^1_{-1}} + 2y\mathscr{D}_{\varphi^0_1}\\ &\quad - (2u_{1,0}y - 2u_{1,1}y^2 - 2u_{2,0}xy + x^3)\mathscr{D}_{\varphi^0_0}.\end{aligned}$$

Remark 6. In the case of evolution equations, existence of a commutative hierarchy means that the initial equation possesses "higher analogs" that are obtained by the action of a recursion operator. In the general case, there exists no well defined action of recursion operators on the equation and thus such a construction is impossible. Nevertheless, one can consider the following alternative scheme: (1) to pass, if possible, to the evolutionary presentation of the equation at hand, (2) to construct "higher analogs" in this presentation and (3) return back to the initial variables. We did not set the question whether such a scheme is invariant (i.e., whether the result is completely determined by the initial equation or depends on the procedure). Probably, an answer to this question will lead to a deeper understanding of integrability for general equations.

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